RELATIONS AMONG WHITNEY SETS, SELF-SIMILAR ARCS AND QUASI-ARCS

BY

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ABSTRACT

We study in this paper some relations among self-similar arcs, Whitney sets and quasi-arcs: we prove that any self-similar arc of dimension greater than 1 is a Whitney set; give a geometric sufficient condition for a self-similar arc to be a quasi-arc, and provide an example of a self-similar arc such that any subarc of it fails to be a t-quasi-arc for any $t \geq 1$, which answers an open question on Whitney sets. We also show that self-similar arcs with the same Hausdorff dimension need not be Lipschitz equivalent.

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1. Introduction

In 1935, Whitney [W1] published his celebrated example of a C^1 function f: $\mathbf{R}^2 \to \mathbf{R}$ critical but not constant along an arc γ of Hausdorff dimension $\log 4/\log 3$. In this example, the image of the critical set of f contains an interval which has positive Lebesgue measure. Why does the above Whitney phenomenon seem to contradict the Morse–Sard Theorem? This is due to the fact that the arc γ is a fractal and f has lower smoothness.

Definition 1: A connected set $E \subset \mathbf{R}^n$ is said to be a Whitney set, if there is a C^1 function f: $\mathbf{R}^n \to \mathbf{R}$ such that f is critical but not constant on E, i.e., the grads $\nabla f|_E \equiv 0$ and $f|_E$ is not a constant.

Remark 1: By an application of the Morse–Sard Theorem, the function f in Definition 1 cannot be sufficiently differentiable (Sard [Sa], 1942).

How can one characterize the Whitney set geometrically? The question was vaguely posed in Whitney's original paper, and can be stated as follows: *Given a function f, how far from rectifiable must a closed connected set be to be a critical set for f on which f is not constant?*

The following two kinds of sets are not Whitney sets:

(1) The set F holding the condition: every pair of points in F is connected by a rectifiable arc lying in F (Whyburn [W], 1929).

(2) The graph G of any continuous function g: $\mathbf{R} \to \mathbf{R}$ (Choquet [C], 1944).

Definition 2: An arc γ is called a t-quasi-arc with $t \geq 1$, if there is a constant λ such that

$$|\gamma(x,y)|^t \leq \lambda|x-y| \quad \text{for any } x, y \in \gamma,$$

where $|\gamma(x,y)|$ is the diameter of $\gamma(x,y)$—the subarc lying between x and y. In particular, a 1-quasi-arc is called a quasi-arc.

Then in 1989, by introducing the above notion of t-quasi-arc, Norton [N1] obtained a sufficient condition for an arc γ to be a Whitney set:

$$\text{If } \gamma \text{ is a } t\text{-quasi-arc with } t < \dim_H \gamma, \text{ then } \gamma \text{ is a Whitney set.}$$

There are therefore two problems raised naturally.

The first is whether or not the t-quasi-arc is also the necessary condition for an arc to be a Whitney set? Here is an open problem posed by Norton [N1]:

Is there an arc γ and a C^1 function f critical but not constant on γ such that, for every subarc η of γ on which f is not constant, η fails to be a t-quasi-arc for any $t \in (1, \infty)$?

The second is: under what condition will self-similar arcs be quasi-arcs, and how to classify self-similar arcs as Falconer and Marsh did for quasi-circles [FM1]. There is a result to this problem:

Two self-similar arcs P and Q are quasi-isometric if and only if $\dim_H(P) = \dim_H(Q)$ (Theorem in [H], p. 116). The statement means that two self-similar arcs are Lipschitz equivalent if and only if they have the same Hausdorff dimension.

About this statement, we will give a counterexample (Theorem 3).

Definition 3: Two sets E and F are said to be nearly Lipschitz equivalent if, for any $a < 1$, there exists a bijection $f_a\colon E \to F$ satisfying

$$c|x-y|^{1/a} \le |f_a(x) - f_a(y)| \le c'|x-y|^a \quad (x, y \in E),$$

for positive constants c and c'. In the category of compact sets, Lipschitz equivalence implies nearly Lipschitz equivalence.

In this paper, we will prove the following results.

THEOREM 1: *Any self-similar arc of Hausdorff dimension greater than 1 is a Whitney set.*

THEOREM 2: *There is a self-similar arc γ with $\dim_H \gamma > 1$ such that every subarc η of γ fails to be a t-quasi-arc for any $t \ge 1$.*

THEOREM 3: *There are self-similar arcs P and Q with $\dim_H(P) = \dim_H(Q)$ such that they are neither Lipschitz equivalent nor nearly Lipschitz equivalent.*

Remark 2: (1) Theorem 1 shows that any self-similar *fractal* arc is a Whitney set. (*Fractal* means a set whose Hausdorff and topological dimension disagree.) Therefore, we obtain another sufficient condition for an arc to be a Whitney set. But the question, how to characterize the Whitney set, remains open.

(2) With an application of Theorem 1 to Theorem 2, we give an affirmative answer to the open problem of Norton.

(3) Theorem 2 says that self-similar arcs may not be quasi-arcs. In Section 5 of this paper, a sufficient condition (Proposition 3) is provided for a self-similar arc to be a quasi-arc. Using this condition, for every $1 < s < 2$, we construct a self-similar planar arc which is a quasi-arc of Hausdorff dimension s.

(4) Theorem 3 gives a counterexample of the conclusion in [H]. In fact, we can construct two self-similar arcs P and Q such that $\dim_H(P) = \dim_H(Q)$, and P is a quasi-arc but Q is not. Since the property of being quasi-arc is invariant under

bi-Lipschitz mapping, we conclude that P and Q are not Lipschitz equivalent. Falconer and Marsh [FM2] pointed out that self-similar sets of Cantor type with the same dimension need not be Lipschitz equivalent, but are nearly Lipschitz equivalent. However, the result of this type does not work for self-similar arcs, as Theorem 3 shows that P and Q are not nearly Lipschitz equivalent.

The rest of the paper is organized as follows: in Section 2, we introduce some preliminaries of self-similar arcs. Section 3 is devoted to the proof of Theorem 1; the main idea is to estimate a probability measure defined on an appropriate Cantor subset of the self-similar arc. Section 4 is devoted to the proof of Theorem 2, that is, we give an example of a self-similar arc which is not a t-quasi-arc. The construction is based on some well-approximable irrational numbers. In Section 5, we give a geometric sufficient condition which guarantees a self-similar arc to be a quasi-arc. The last section is devoted to the proof of Theorem 3.

2. Preliminaries

A mapping S: $\mathbf{R}^n \to \mathbf{R}^n$ is called a contractive similitude if there is a ratio ρ with $0 < \rho < 1$ such that, for any $x, y \in \mathbf{R}^n$,

$$|S(x) - S(y)| = \rho|x - y|.$$

Suppose $\mathbf{S} := \{S_i, \ldots, S_m\}$ is a family of similitudes with contraction ratios $\rho_1, \ldots, \rho_m$. Then there exists a unique compact set E such that

$$E = \bigcup_{i=1}^{m} S_i(E);$$

the set E is called the self-similar set generated by $\mathbf{S}$ [Hu].

Let $\dim_H(\cdot)$ denote the Hausdorff dimension, and $\mathcal{H}^s$ the Hausdorff measure of dimension s. Throughout the paper, an arc means a homeomorphic image of the unit interval [0,1].

Definition 4: An arc γ is called a self-similar arc, if γ is generated by a family of contractive similitudes $\mathbf{S} = \{S_i\}_{1\le i\le m}$ satisfying

(1) $S_i(\gamma) \cap S_j(\gamma)$ is a singleton for $|i-j| = 1$;

(2) $S_i(\gamma) \cap S_j(\gamma) = \emptyset$ for $|i-j| > 1$.

We denote by $\gamma = \gamma(\mathbf{S})$ the self-similar arc generated by the family of contractive similitudes $\mathbf{S} = \{S_i\}_{1\le i\le m}$.

Remark 3: Let $s := \dim_H \gamma$; then by the definition of a self-similar arc, we get

$$(*) \qquad \mathcal{H}^s[S_i(\gamma) \cap S_j(\gamma)] = 0 \quad \text{for } i \neq j.$$

By [Sc], the family $\mathbf{S}$ fulfills the open set condition. Consequently, $s = \dim_H \gamma$ is the solution of the equation

$$\sum_{i=1}^{m} (\rho_i)^s = 1,$$

where $\{\rho_i\}_{i=1}^m$ are contraction ratios of $\{S_i\}_{i=1}^m$; see [Hu].

In general, the formula $(*)$ cannot guarantee that γ is an arc. For example, a fractal including three similar branches which share one endpoint is not an arc.

Let $\Sigma = \{1, \ldots, m\}$ be the finite set of m symbols, let $\Sigma^*(n) = \{i_1 \cdots i_n | i_t \in \Sigma$ for all $t\}$ be the set of the sequences of length n, and $\Sigma^* = \bigcup_{n \geq 0} \Sigma^*(n)$ the set of the sequences of finite length.

For any $i^* = i_1 i_2 \cdots i_k \in \Sigma^*(k)$, write $S_{i^*} = S_{i_1} \circ \cdots \circ S_{i_k}$; then the contraction ratio of S_{i^*} is $\rho(i^*) = \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k}$.

The following lemma shows that by selecting an appropriate family of similitudes, the endpoints of a self-similar arc are fixed points of two similitudes. This lemma is useful for Proposition 3 in Section 5.

LEMMA 1: *Suppose γ is a self-similar arc with endpoints a, b. Then there exists a family of similitudes $\mathbf{S} = \{S_i\}_{1 \leq i \leq N}$ such that $\gamma = \gamma(\mathbf{S})$ and $a = S_1(a)$, $b = S_N(b)$.*

Proof: Suppose $\gamma = \gamma(\mathbf{T})$ with similitudes $\mathbf{T} = \{T_i\}_{i=1}^m$, and $a \in T_1(\gamma)$, $b \in T_m(\gamma)$.

By the definition of self-similar-arc, we see that a is either $T_1(a)$ or $T_1(b)$. Similarly, $b = T_m(a)$ or $T_m(b)$. There are therefore only 4 possible cases as follows.

$$\begin{aligned}
&\text{Case 1.} \quad a = T_1(a), \quad b = T_m(b);\\
&\text{Case 2.} \quad a = T_1(a), \quad b = T_m(a);\\
&\text{Case 3.} \quad a = T_1(b), \quad b = T_m(a);\\
&\text{Case 4.} \quad a = T_1(b), \quad b = T_m(b).
\end{aligned}$$

Let $\mathbf{S} = \{T_{i_1 i_2 i_3 i_4} : 1 \leq i_1, i_2, i_3, i_4 \leq m\}$; then $\gamma = \gamma(\mathbf{S})$. Hence the conclusion of the lemma follows by rearranging the indices of the elements of $\mathbf{S}$. For example, in the case 2, set $S_1 = T_{1111}$ and $S_{m^4} = T_{m11m}$; then a and b are fixed points of S_1 and S_{m^4} respectively. ∎

3. Proof of Theorem 1

We need a special case of the Whitney Extension Theorem which is stated as follows [W2].

LEMMA 2: *Suppose $E \subset \mathbf{R}^n$ is compact and $f\colon E \to \mathbf{R}$ is a real function. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in E$ with $0 < |x - y| < \delta$,*

$$|f(x) - f(y)|/|x - y| < \varepsilon,$$

then there is a C^1 extension $\bar{f}\colon \mathbf{R}^n \to \mathbf{R}$ of f such that $\bar{f}|_E = f$ and $\nabla \bar{f}|_E \equiv 0$.

Now we prove Theorem 1 by the following several steps.

STEP 1: Find $n \in \mathbb{N}$ and a subset $A \subset \Sigma^*(n)$ such that

(H1) $$S_{a_1^*}(\gamma) \cap S_{a_2^*}(\gamma) = \emptyset \text{ for any distinct pair } a_1^*, a_2^* \in A;$$

(H2) $$\sum_{a^* \in A} \rho(a^*)^s = 1 \text{ for some constant } s > 1;$$

(H3) $$\text{for any } a^* \in A, \text{ the endpoints of } \gamma \text{ do not belong to } S_{a^*}(\gamma).$$

To get (H1), take $s^- \in (1, \dim_H \gamma)$; since $\sum_{a^* \in \Sigma^*(n)} \rho(a^*)^{\dim_H \gamma} = 1$, we have

$$\begin{aligned}\sum_{a^* \in \Sigma^*(n)} \rho(a^*)^{s^-} \geq & \left[\sum_{a^* \in \Sigma^*(n)} \rho(a^*)^{\dim_H \gamma} \right] / [\max_i \rho_i]^{n(\dim_H \gamma - s^-)} \\ = & 1/[\max_i \rho_i]^{n(\dim_H \gamma - s^-)},\end{aligned}$$

which goes to infinity as n goes to infinity. Thus for n large enough, we can divide $\Sigma^*(n)$ into two subsets A^1 and A^2 such that for $i = 1, 2$, and any distinct pair $a^*, b^* \in A^i$, $S_{a^*}(\gamma) \cap S_{b^*}(\gamma) = \emptyset$.

Without loss of generality, we assume that

$$\sum_{a^* \in A^1} \rho(a^*)^{s^-} > 2.$$

Delete at most two elements of A^1 containing the endpoint of γ, and denote by A the obtained set. Then $A \subset \Sigma^*(n)$ and, for n large enough, we have

$$\sum_{a^* \in A} \rho(a^*)^{s^-} \geq \sum_{a^* \in A^1} \rho(a^*)^{s^-} - 2[\max_i \rho(i)]^{ns^-} > 1,$$

thus we can find $s \in (s^-, \dim_H \gamma)$ such that $\sum_{a^* \in A} \rho(a^*)^s = 1$.

By the above discussion, we get (H_2) and (H_3).

STEP 2: Construct a function f on γ.

Let μ be a mass distribution satisfying

$$\mu(S_{a_1^*} \circ S_{a_2^*} \circ \cdots \circ S_{a_k^*}(\gamma)) = \begin{cases} [\prod_{i=1}^k \rho(a_i^*)]^s & \text{if } a_i^* \in A \text{ for every } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu(\gamma) = 1$ from the fact that $\sum_{a^* \in A} \rho(a^*)^s = 1$, so μ is a probability measure on γ.

Fix one endpoint e of the arc γ. We define a function $f: \gamma \to \mathbf{R}$ in the following way: for any $x \in \gamma$,

$$f(x) = \mu[\gamma(e,x)], \tag{1}$$

where $\gamma(e,x)$ is the subarc of γ between e and x.

STEP 3: Using Lemma 2, estimate $\frac{|f(x)-f(y)|}{|x-y|}$ for $x, y \in \gamma$.

Since the self-similar arc γ can be generated also by $\{S_{a^*}\}_{a^* \in \Sigma^*(n)}$, consider now a symbolic system generated by the finite set $\Sigma^*(n)$. Set

$$\Lambda^* = \{a_1^* a_2^* \cdots a_k^* |\ a_i^* \in \Sigma^*(n), k \geq 1, 1 \leq i \leq k\},$$
$$\Lambda^*(A) = \{a_1^* a_2^* \cdots a_k^* |\ a_i^* \in A, k \geq 1, 1 \leq i \leq k\}.$$

On Λ^*, we define a partial order as follows:

$$a_1^* a_2^* \cdots a_k^* a_{k+1}^* \cdots a_l^* \prec a_1^* a_2^* \cdots a_k^*,$$

where $a_i^* \in \Sigma^*(n)$.

Given distinct points $x, y \in \gamma$, suppose $c^* \in \Lambda^*$ is a minimal element under the above partial order such that $x, y \in S_{c*}(\gamma)$. Then there are two distinct sequences $d_1^*, d_2^* \in \Sigma^*(n)$ such that

$$x \in S_{c*}[S_{d_1^*}(\gamma)], \quad y \in S_{c*}[S_{d_2^*}(\gamma)].$$

Now we distinguish two cases.

CASE I: $S_{d_1^*}(\gamma) \cap S_{d_2^*}(\gamma) = \emptyset$.

From the definition of f, we have

$$|f(x) - f(y)| \leq \mu[S_{c*}(\gamma)] = \rho(c^*)^s. \tag{2}$$

Let $\delta_1 > 0$ be the least distance between any two disjoint subarcs $S_{b_1^*}(\gamma)$ and $S_{b_2^*}(\gamma)$ with $b_1^*, b_2^* \in \Sigma^*(n)$. Then

$$\begin{aligned} |x-y| &\geq d(S_{c*}[S_{d_1^*}(\gamma)], S_{c*}[S_{d_2^*}(\gamma)]) \\ &\geq \rho(c^*) d(S_{d_1^*}(\gamma), S_{d_2^*}(\gamma)) \geq \rho(c^*)\delta_1; \end{aligned} \tag{3}$$

from (2) and (3),

$$\frac{|f(x)-f(y)|}{|x-y|} \leq (\delta_1)^{-1} \cdot \rho(c^*)^{s-1}. \tag{4}$$

Case II: $S_{d_1^*}(\gamma) \cap S_{d_2^*}(\gamma) \neq \emptyset$.

In this case, suppose the singleton $S_{c*}[S_{d_1^*}(\gamma)] \cap S_{c*}[S_{d_2^*}(\gamma)] = \{z\}$.

From the definition of f,

$$|f(x)-f(y)| = 0 \tag{5}$$

holds in the following two situations:

(D1) $c^* \notin \Lambda^*(A)$ (since $\mu[S_{c^*}(\gamma)] = 0$);

(D2) $c^* \in \Lambda^*(A)$, and neither sequence d_1^*, d_2^* belongs to A (since $\mu[S_{c^*d_1^*}(\gamma)] = \mu[S_{c^*d_2^*}(\gamma)] = 0$).

In the situations other than above, we suppose $c^* \in \Lambda^*(A)$ and at least one of d_i belongs to A. From (H1) of the construction of the set A, one may assume $d_1^* \in A$, $d_2^* \notin A$. Assume furthermore $x \in S_{c*}[S_{d_1^*}(S_{e^*}(\gamma))]$, $z \in S_{c*}[S_{d_1^*}(S_{f^*}(\gamma))]$ with $e^*, f^* \in \Sigma^*(n)$.

If $e^* = f^*$, since z is an endpoint of $S_{c*}[S_{d_1^*}(\gamma)]$, it follows from (H3) that

$$|f(x)-f(y)| = |f(x)-f(z)| = 0. \tag{6}$$

If $e^* \neq f^*$, then $|f(x)-f(y)| \leq \mu[S_{c*}(\gamma)] = \rho(c^*)^s$. Let $\delta_2 > 0$ be the least distance between any two disjoint subarcs $S_{b_1^*}[S_{d^*}(\gamma)]$ and $S_{b_2^*}(\gamma)$)with $b_1^*, b_2^*, d^* \in \Sigma^*(n)$. Then

$$|x-y| \geq \rho(c^*) d(S_{d_1^*}(S_{e^*}(\gamma)), S_{d_2^*}(\gamma)) \geq \rho(c^*)\delta_2, \tag{7}$$

so

$$\frac{|f(x)-f(y)|}{|x-y|} \leq (\delta_2)^{-1} \cdot \rho(c^*)^{s-1}. \tag{8}$$

From (4)–(8), for any distinct points $x, y \in \gamma$, we have

$$\frac{|f(x)-f(y)|}{|x-y|} \leq \tau |x-y|^{s-1},$$

where τ is a positive constant.

Since $s > 1$, by using Lemma 2, we can extend f to a C^1 function $\bar{f}\colon \mathbf{R}^n \to \mathbf{R}$ such that

$$\bar{f}|_\gamma = f \quad \text{and} \quad \nabla \bar{f}|_\gamma \equiv 0,$$

i.e., the self-similar arc γ is a Whitney set. ∎

4. Proof of Theorem 2

For the proof of Theorem 2, we need the following Lemma of Diophantine approximation.

LEMMA 3: *Let x be an irrational number with the following property: for any fixed positive integer m_0, there exist infinitely many integer pairs $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that*

$$\left|x - \frac{m+m_0}{n}\right| \le e^{-n^2}\cdot\frac{1}{n^2}.$$

Then the set of irrational numbers with above property is dense in $\mathbb{R}^+$.

Proof: Define the sequence $\{a_n\}_n \ge 1$ by

$$a_n = \begin{cases} e^{-n^2}/n & \text{if } n = 2^k \text{ for } k \in \mathbb{N}\cup\{0\},\\ 0 & \text{if } 2^k < n < 2^{k+1} \text{ for some } k \in \mathbb{N};\end{cases}$$

then $\sum_{n\ge1} a_n < \infty$. Set $Q = 2^k$ ($k \in \mathbb{N}$); then there exist infinitely many integers Q such that

$$\sum_{n\le Q} a_n \le 2\sum_{n\le Q} a_n\frac{\varphi(n)}{n},$$

where the Euler function $\varphi(n) = \#\{1 \le m \le n\text{: g.c.d. }(m,n) = 1\}$. The above inequality holds since $\varphi(2^k) = 2^{k-1}$. Then from the Duffin–Schaeffer Theorem [DS], for *almost all* $x \in \mathbb{R}$, there are infinitely many pairs of integers (n,m) satisfying $|nx - m| < a_n$ with $(n,m) = 1$. This yields the conclusion of the lemma. ∎

To prove Theorem 2, we will construct a self-similar arc illustrated as in Figure 1: $\Delta A_0A_3A_7$ is an isosceles triangle with $|A_0A_7| = 1$, the base angle θ is small enough. Let K be the convex closure of this triangle.

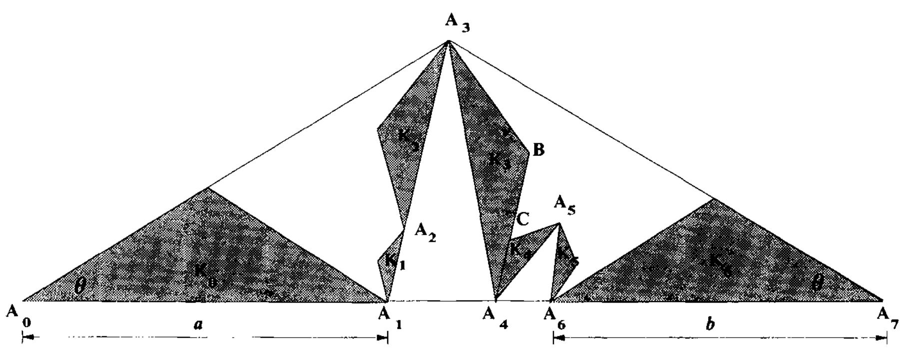

Figure 1.

Consider seven similitudes S_i: $K \to K_i$ (see Figure 1) which satisfy

$$S_i(A_0) = A_i, \quad S_i(A_7) = A_{i+1}, \quad 0 \le i \le 6,$$

and:

(C1) $|A_0A_1| = a$, $|A_6A_7| = b$ such that $\log b/\log a$ is a positive irrational number fulfilling the condition of Lemma 3; this is possible since such numbers are dense in $\mathbb{R}^+$ by Lemma 3;

(C2) $|A_4A_5| = a^{m_0}|A_3A_4|$, where the integer $m_0 > 0$ is large enough;

(C3) $K_3 \cap K_4 = A_4C \subset A_4B$.

Since θ is small enough and $\{K_i\}_i$ are similar to K, we see that from Figure 1

$$\bigcup_{i=0}^{6} int(K_i) \subset int(K) \quad \text{and} \quad int(K_i) \cap int(K_j) = \emptyset \quad \text{for } i \ne j,$$

where $int(A)$ denotes the interior of the set A. Let E denote the self-similar set generated by the similitudes $\{S_i\}_{0 \le i \le 6}$.

Remark 4: Although the intersection of K_3 and K_4 is the segment A_4C, the intersection of $K_3 \cap E$ and $K_4 \cap E$ consists of just one point A_4.

Now we are going to show that the set E is a self-similar arc; in fact, we will prove that E can be parameterized by the following homeomorphism γ: $[0,1] \to E$:

$$\gamma\left(\sum_{n=1}^{\infty} \frac{a_n}{7^n}\right) = \lim_{n\to\infty} S_{a_1} \circ S_{a_2} \circ \cdots \circ S_{a_n}(E) \in \mathbf{R}^2 \quad (0 \le a_n \le 6).$$

PROPOSITION 1: *E is an arc.*

Proof: It suffices to show

$$\begin{cases} S_i(E) \cap S_j(E) = \emptyset & \text{if } |i-j| \ge 2, \\ S_i(E) \cap S_{i+1}(E) = \{A_{i+1}\} & \text{if } 0 \le i \le 6. \end{cases}$$

Since $E \subset K$, $S_i(E) \subset S_i(K) = K_i$.

(1) If $|i-j| \ge 2$, then $K_i \cap K_j = \emptyset$, which yields

$$S_i(E) \cap S_j(E) = \emptyset.$$

(2) If $i \ne 3$, then $K_i \cap K_{i+1} = \{A_{i+1}\}$ (see Figure 1), so

$$S_i(E) \cap S_{i+1}(E) = \{A_{i+1}\}.$$

(3) Now consider $i = 3$. In this case $K_3 \cap K_4$ is the line segment A_4C; notice that A_4B and A_4C are the images of A_3A_7 and A_0A_3 under the mappings S_3 and S_4 respectively. Therefore,

$$\begin{aligned} S_3(E) \cap S_4(E) =& (E \cap BA_4) \cap (E \cap A_4C) \\ =& S_3(A_3A_7 \cap E) \cap S_4(A_0A_3 \cap E). \end{aligned}$$

From the structure of E, we see that

$$A_0A_3 \cap E = \{A_0, A_3, \ldots, S_0^m(A_3), \ldots\} = \{A_0\} \cup \{S_0^m(A_3)\}_{m\geq 0}.$$

On the other hand, since $S_0(x) = (1-a)A_0 + ax$, we have $S_0^m(A_3) = (1-a^m)A_0 + a^m A_3$ $(m \geq 0)$, thus

$$A_0A_3 \cap E = \{A_0\} \cup \{(1-a^m)A_0 + a^m A_3\}_{m\geq 0}.$$

In the same way we obtain

$$A_3A_7 \cap E = \{A_7\} \cup \{(1-b^n)A_7 + b^n A_3\}_{n\geq 0}.$$

From the above discussion, we get

$$S_3(E) \cap S_4(E) = \{A_4\} \cup G,$$

where

$$G = \{(1-a^m)A_4 + a^m C\}_{m\geq 0} \cap \{(1-b^n)A_4 + b^n B\}_{n\geq 0}.$$

We conclude that G is empty. Otherwise, suppose $x \in G$; then there exist integers $m_1, n_1 \geq 0$ such that $|xA_4| = |A_4B|b^{n_1} = |A_4C|a^{m_1}$. But from the condition (C2) of the construction of E, we have

$$\begin{aligned} |A_4B| =& |A_3A_4| \cdot |A_3A_7|, \\ |A_4C| =& |A_4A_5| \cdot |A_0A_3| = a^{m_0}|A_3A_4| \cdot |A_3A_7|, \end{aligned}$$

which imply

$$a^{m_0+m_1} = b^{n_1}$$

with $m_0 > 0, m_1, n_1 \geq 0$, and $n_1 \neq 0$ obviously. Therefore $\log b/\log a = (m_1 + m_0)/n_1$ is a rational number, which contradicts the choice of the irrationality of $\log b/\log a$. We thus complete the proof of the proposition. ∎

PROPOSITION 2: *Any subarc of E fails to be a t-quasi-arc for any $t \geq 1$.*

Proof: Because of the self-similarity of E, it suffices to show that E is not a t-quasi-arc for any $t \geq 1$. Suppose the conclusion is not true; then E is a t-quasi-arc for some $t \geq 1$, so there exists a positive constant λ such that

$$|\gamma(x,y)|^t \leq \lambda |x-y|, \quad \forall x, y \in \gamma.$$

Consequently,

$$t \geq \limsup_{|x-y|\to 0} \frac{\log|x-y|}{\log|\gamma(x,y)|}. \tag{9}$$

From Lemma 3, there exist infinitely many pairs of integers $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that

$$\left|\frac{\log b}{\log a} - \frac{m+m_0}{n}\right| \leq e^{-n^2}\cdot\frac{1}{n^2},$$

which implies

$$|(m+m_0)\log a - n\log b| \leq e^{-n^2}\cdot\frac{|\log a|}{n}. \tag{10}$$

Fix a pair (m,n) satisfying (10), and take

$$x = (1-a^m)A_4 + a^m C, \quad y = (1-b^n)A_4 + b^n B.$$

It follows from the proof of Proposition 1 that $x, y \in E$. Moreover (see Figure 1),

$$\begin{aligned}
&|x-y| = |\,|A_4 - x| - |A_4 - y|\,| = |\,a^m|A_4C| - b^n|A_4B|\,| \\
=&\left|a^m\frac{|A_4C|}{|A_4B|} - b^n\right|\cdot|A_4B| = \left|a^m\frac{|A_4A_5|}{|A_4A_3|} - b^n\right|\cdot|A_4B| \\
=&|a^{m_0+m} - b^n|\cdot|A_4B|,
\end{aligned}$$

the last equality being due to (C_2).

On the other hand, since $A_4 \in \gamma(x,y)$, $|\gamma(x,y)| \geq |A_4 - y| = b^n|A_4B|$, thus

$$\frac{\log|x-y|}{\log|\gamma(x,y)|} \geq \frac{\log(|a^{m+m_0} - b^n|^{-1}) - \log|A_4B|}{n|\log b| - \log|A_4B|}. \tag{11}$$

Notice that $|e^t - 1| \leq 2|t|$ if $|t|$ is small enough, therefore

$$\log(|e^t - 1|^{-1}) \geq \log(|t|^{-1}/2).$$

Let $t = (m+m_0)\log a - n\log b$; then from (10), $|t|$ is small if n is large enough. Hence

$$\begin{aligned}\log(|a^{m+m_0} - b^n|^{-1}) &= \log(|a^{m+m_0}/b^n - 1|^{-1}) + n|\log b|\\ &= \log(|e^{(m+m_0)\log a - n\log b} - 1|^{-1}) + n|\log b|\\ &\geq \log\left(\left|e^{-n^2}\cdot\frac{|\log a|}{n}\right|^{-1}/2\right) + n|\log b|\\ &\geq n^2 + n|\log b|,\end{aligned}$$

which yields from (11)

$$\frac{\log|x-y|}{\log|\gamma(x,y)|} \geq \frac{n^2 + n|\log b| - \log|A_4B|}{n|\log b| - \log|A_4B|} \geq \alpha n$$

for some positive constant α. Since there are infinitely many pairs (m,n) satisfying (10), it follows from (9) that $t \geq +\infty$. The proposition follows from this contradiction. ∎

Theorem 2 follows from Propositions 1 and 2.

5. Condition for self-similar arcs to be quasi-arcs

In this section, we will give a sufficient condition such that a self-similar arc is a quasi-arc.

Suppose that the self-similar arc η is generated by the contractive similitudes $\{S_i\}_{i=1}^m$ with $S_i(\eta)\cap S_{i+1}(\eta) = \{A_i\}(i = 1,\ldots,m-1)$. We choose always the angle $0 \leq \angle xA_iy \leq \pi$ whenever $x\in S_i(\eta)$ and $y \in S_{i+1}(\eta)$.

PROPOSITION 3: *Suppose that there is a constant $\theta^* > 0$ such that the angle $\angle xA_iy \geq \theta^*$ whenever $x\in S_i(\eta)$, $y\in S_{i+1}(\eta)$ $(1\leq i\leq m-1)$. Then η is a quasi-arc.*

Remark 5: (1) The arc is not restricted on the plane, since $\angle xA_iy$ may be the angle in the space of higher dimension.

(2) We may suppose $\theta^* \leq \pi/2$.

(3) Notice that in the example of the last section, we have $\angle BA_4C = 0$, which doesn't satisfy the condition of the proposition.

Proof: By Lemma 1, without loss of generality we assume that two endpoints A_0, A_m of η are fixed points of S_1 and S_m respectively. As usual, for any sequence $i_1\cdots i_k \in \{1,\ldots,m\}^k$, denote $\eta_{i_1\cdots i_k} = (S_{i_1\cdots i_k})(\eta)$.

Suppose $x, y\in\eta$ with $x\neq y$, and suppose $i_1\cdots i_k$ is the sequence such that $x, y \in \eta_{i_1\cdots i_k}$ but for any i_{k+1}, $\eta_{i_1\cdots i_k i_{k+1}}$ contains at most one of x and y. Let $\bar{x}$

$= (S_{i_1\cdots i_k})^{-1}(x)$ and $\bar{y} = (S_{i_1\cdots i_k})^{-1}(y)$. Then there exist $i \neq j$ such that $\bar{x} \in \eta_i$ and $\bar{y} \in \eta_j$. Thus we have

$$\frac{|\eta(x,y)|}{|x-y|} = \frac{[\prod_{j=1}^{k} \rho_{i_j}] \cdot |\eta(\bar{x},\bar{y})|}{[\prod_{j=1}^{k} \rho_{i_j}] \cdot |\bar{x}-\bar{y}|} = \frac{|\eta(\bar{x},\bar{y})|}{|\bar{x}-\bar{y}|}. \tag{12}$$

For estimating $\frac{|\eta(\bar{x},\bar{y})|}{|\bar{x}-\bar{y}|}$, we distinguish two cases.

CASE 1: $|j-i| > 1$.

In this case, the subarcs η_i, η_j are disjoint. Since $|\eta(\bar{x},\bar{y})| \leq |\eta|$, we have

$$|\bar{x}-\bar{y}| \geq d(\eta_i,\eta_j) \geq \min_{|i_1-i_2|>1} d(\eta_{i_1},\eta_{i_2}) := D > 0.$$

From (12), we have

$$|\eta(\bar{x},\bar{y})| \leq (D^{-1}|\eta|) \cdot |\bar{x}-\bar{y}|. \tag{13}$$

CASE 2: $j-i=1$.

Let A be the common point of η_i and η_j. Since A_0 and A_m are the endpoints of the arc by assumption, we have either $A = S_j(A_0)$ or $S_j(A_m)$. Without loss of generality, assume $A = S_j(A_0)$. Since A_0 is the fixed point of S_1, for any $n \geq 0$, $A = S_j(S_1)^n(A_0) \in S_j(S_1)^n(\eta)$. Hence for the point $\bar{y} \neq A$, there exist $n_0 \geq 0$ and $i_0 \neq 1$ such that $\bar{y} \in S_j(S_1)^{n_0}S_{i_0}(\eta)$. Notice that $\eta(A,\bar{y}) \subset S_j(S_1)^{n_0}(\eta)$. Then

$$|\eta(A,\bar{y})| \leq |S_j(S_1)^{n_0}(\eta)| = \rho_j(\rho_1)^{n_0}|\eta|. \tag{14}$$

On the other hand, since $A = S_j(S_1)^{n_0}(A_0)$ and $\bar{y} \in S_j(S_1)^{n_0}S_{i_0}(\eta)$,

$$\begin{aligned} |A-\bar{y}| &\geq d[A, S_j(S_1)^{n_0}S_{i_0}(\eta)] \\ &\geq d[S_j(S_1)^{n_0}(A_0), S_j(S_1)^{n_0}S_{i_0}(\eta)] \\ &\geq \rho_j(\rho_1)^{n_0} d[A_0, S_{i_0}(\eta)] \geq \rho_j(\rho_1)^{n_0}\delta^* \end{aligned} \tag{15}$$

where $\delta^* := \min[\min_{t>1} d(A_0,\eta_t), \min_{t<m} d(A_m,\eta_t)] > 0$.

From (14) and (15), if $\bar{y} \neq A$ or $\bar{y} = A$, then

$$|\eta(A,\bar{y})| \leq (\delta^*)^{-1}|\eta| \cdot |\bar{y}-A|. \tag{16}$$

Notice that if $\bar{y} = A$, the above inequality holds trivially.

By the same way, we get

$$|\eta(\bar{x},A)| \leq (\delta^*)^{-1}|\eta| \cdot |\bar{x}-A|. \tag{17}$$

Since $\angle\bar{x}A\bar{y} \geq \theta^* > 0$ by the hypotheses, we get

$$\begin{aligned}|\bar{x}-\bar{y}|^2 &= |\bar{x}-A|^2 + |\bar{y}-A|^2 - 2\cos(\angle\bar{x}A\bar{y})\cdot|\bar{x}-A||\bar{y}-A|\\ &\geq (1-\cos\theta^*)(|\bar{x}-A|^2+|\bar{y}-A|^2) + \cos\theta^*(|\bar{x}-A|-|\bar{y}-A|)^2\\ &\geq \frac{1-\cos\theta^*}{2}[2\cdot(|\bar{x}-A|^2+|\bar{y}-A|^2)] \quad (\text{using } \cos\theta^*>0)\\ &\geq \sin^2(\theta^*/2)(|\bar{x}-A|+|\bar{y}-A|)^2 \quad (\text{using } 2(c^2+d^2)\geq(c+d)^2),\end{aligned}$$

which yields

$$|\bar{x}-\bar{y}| \geq \sin(\theta^*/2)\cdot(|\bar{x}-A|+|\bar{y}-A|). \tag{18}$$

From (16), (17) and (18), we have

$$\begin{aligned}|\bar{x}-\bar{y}| &\geq \sin(\theta^*/2)\cdot[|\bar{x}-A|+|\bar{y}-A|]\\ &\geq k'[|\eta(\bar{x},A)|+|\eta(\bar{y},A)|] \geq k'|\eta(\bar{x},\bar{y})|,\end{aligned} \tag{19}$$

where constant $k' > 0$.

From (13) and (19), we prove that η is a quasi-arc. ∎

Remark 6: The classical von Koch curve is a quasi-arc; in fact, we can take $\theta^* = \pi/3$ in this case.

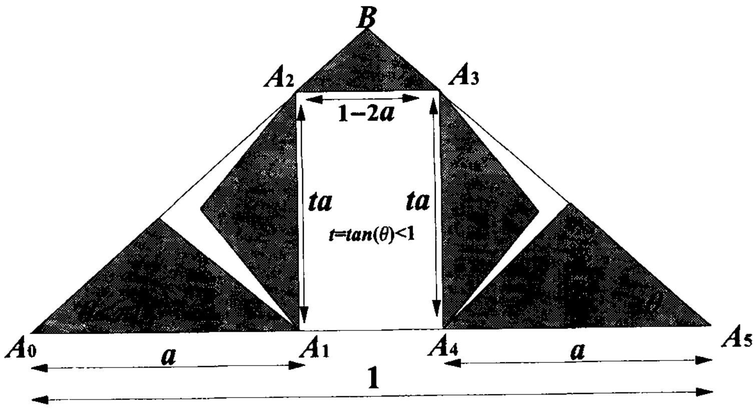

Figure 2.

In Figure 2, $A_iA_{i+1}\perp A_{i+1}A_{i+2}$ for $0 \leq i \leq 3$, $\theta < \pi/4$, $|A_0A_5| = 1$, $0 < a < 1/2$, and $t = \tan(\theta) < 1$. In the isosceles triangle ΔA_0BA_5, the structure of five small similar triangles provides a self-similar arc. It follows from

Proposition 3 that it is also a quasi-arc. The Hausdorff dimension s of the arc satisfies the equation

$$2[a^s + t^s a^s] + (1 - 2a)^s = 1,$$

which implies $s \to 1$ as $a \to 0$, and $s \to 2$ as $a \to 1/2$, $t \to 1$. Consequently, we have the following corollary.

COROLLARY 1: *For any s with $1 < s < 2$, there is a self-similar quasi-arc of Hausdorff dimension s.*

6. Proof of Theorem 3

Suppose Q is a self-similar planar arc such that any subarc of Q fails to be a t-quasi-arc for any t. By Corollary 1, we can select another self-similar quasi-arc P with $\dim_H P = \dim_H Q$. We will show that P and Q are not Lipschitz equivalent or nearly Lipschitz equivalent. In fact, if P and Q are Lipschitz equivalent or nearly Lipschitz equivalent, then there exists a bijection $f_a\colon P \to Q$ with $a \leq 1$ such that for any $x, y \in P$,

$$c|x - y|^{1/a} \leq |f_a(x) - f_a(y)| \leq c'|x - y|^a,$$

where c and c' are positive constants. Since P is a quasi-arc, there is a constant $\lambda > 0$ such that for any $x, y \in P$, $|P(x, y)| \leq \lambda|x - y|$. Notice that $Q(f_a(x), f_a(y)) = f_a(P(x, y))$; we have

$$\begin{aligned}|Q(f_a(x), f_a(y))|^{1/a^2} =&|f_a(P(x,y)|^{1/a^2} \leq (c')^{1/a^2}|P(x,y)|^{1/a}\\ \leq&(c')^{1/a^2}\lambda^{1/a}|x-y|^{1/a} \leq [(c')^{1/a^2}\lambda^{1/a}/c]\cdot|f_a(x) - f_a(y)|\end{aligned}$$

whenever $f_a(x), f_a(y) \in Q$. That shows Q is a $1/a^2$-quasi-arc, which contradicts the choice of Q.

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